RALEIGH WAVES IN A NONHOMOGENEOUS ELASTIC HALF-SPACE

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We are considering a displacement field in the isotropic elastic medium which occupies a half-plane whose Lamé's coefficients and density are arbitrary smooth functions of the depth. The boundary is assumed to be stress-free. An exact solution is constructed in the form of a monochromatic wave, which is then investigated asymptotically for high frequencies. It turns out that there exists a solution analogous to the ordinary Rayleigh wave in the homogeneous elastic half-plane. An expression for the correction term in the asymptotic representation of the dispersion of the phase velocity is obtained.

Let the half-plane $-\infty < x < +\infty$, $z \geqslant 0$ be occupied by an elastic medium with Lamé's coefficients $\lambda(z)$, $\mu(z)$ and density $\rho(z)$, which are sufficiently smooth functions of the depth z. It is required to find the displacement vector $\mathbf{u}(x, z, t) = (u_x(x, z, t), u_z(x, z, t))$, which for $z \geqslant 0$, $t \geqslant 0$ satisfies the equations of the dynamic theory of elasticity

$$L\mathbf{u} = 0 \qquad \left(L = -\rho I \frac{\partial^2}{\partial t^2} + M + N\right)$$

$$M = \begin{Vmatrix} \nu \dfrac{\partial^2}{\partial x^2} + \mu \dfrac{\partial^2}{\partial z^2} & (\nu - \mu) \dfrac{\partial^2}{\partial x \partial z} \\ (\nu - \mu) \dfrac{\partial^2}{\partial x \partial z} & \nu \dfrac{\partial^2}{\partial z^2} + \mu \dfrac{\partial^2}{\partial x^2} \end{Vmatrix} \qquad (\nu = \lambda + 2\mu) \tag{1}$$

$$N = \begin{Vmatrix} \mu' \dfrac{\partial}{\partial z} & \mu' \dfrac{\partial}{\partial x} \\ \lambda' \dfrac{\partial}{\partial x} & \nu' \dfrac{\partial}{\partial z} \end{Vmatrix}, \qquad I = \begin{Vmatrix} 1, & 0 \\ 0, & 1 \end{Vmatrix}$$

and the condition of the stress-free boundary

$$Pu = 0 \quad \text{for } z = 0, \qquad P = \begin{Vmatrix} \mu \frac{\partial}{\partial z} & \mu \frac{\partial}{\partial x} \\ \lambda \frac{\partial}{\partial x} & \nu \frac{\partial}{\partial z} \end{Vmatrix} \tag{2}$$

We will look for a particular solution of the problem (1) to (2) in the form

$$\mathbf{u}(x, z, t; k) = e^{ikx - i\zeta t}\,\mathbf{G}(z, k, \zeta) \tag{3}$$

which is decreasing with depth. Here $k > 0$, ζ is a complex parameter. The substitution of (3) into (1) and (2) yields for $\mathbf{G}$ the system of equations

$$\frac{d^2}{dz^2}\mathbf{G} + ikA\frac{d}{dz}\mathbf{G} + (ik)^2 B\mathbf{G} + C\frac{d}{dz}\mathbf{G} + ikD\mathbf{G} = 0 \tag{4}$$

and the boundary condition

$$\frac{d}{dz}\mathbf{G} + ikE\mathbf{G} = 0, \qquad z = 0 \tag{5}$$

where the elements of matrices A, B, C, D, E of the second order depend on z and σ ($\sigma = \zeta k^{-1}$).

Let $\mathbf{G}^{(p)}(z, k, \sigma)$ and $\mathbf{G}^{(s)}(z, k, \sigma)$ be two linearly independent solutions of the system (4), which decrease with depth. We now choose the constants α and β so as to have the solution $\alpha\mathbf{G}^{(p)} + \beta\mathbf{G}^{(s)}$ of the system (4) satisfy the condition (5). It is easy to see that α and β are determined from the system of equations

$$D_p(k, \sigma)\,\alpha + D_s(k, \sigma)\,\beta = 0, \qquad E_p(k, \sigma)\,\alpha + E_s(k, \sigma)\,\beta = 0 \tag{6}$$

where

$$D_p(k, \sigma) = \left(\frac{d}{dz}\, G_x^{(p)} + ikG_z^{(p)}\right)\Big|_{z=0} \quad \left(p = \frac{\mu}{\nu}, \quad 0 < p < \frac{1}{2}\right)$$
$$E_p(k, \sigma) = \left(\frac{d}{dz}\, G_z^{(p)} + ik(1 - 2p)\, G_x^{(p)}\right)\Big|_{z=0} \tag{7}$$

The expressions for D_s and E_s are analogous to the above. Obviously, to the non-zero solutions of the problem (4) and (5) there correspond in the complex plane σ the zeros of the function

$$\Delta(k, \sigma) = D_p(k, \sigma)\,E_s(k, \sigma) - E_p(k, \sigma)\,D_s(k, \sigma) \tag{8}$$

and *vice versa*, to each solution $\sigma_*(k)$ of the equation $\Delta(k, \sigma) = 0$ there corresponds a solution of the problems (1) and (2) of the form

$$u_*(x, z, t; k) = e^{ik(x-\sigma_*(k)t)}\,[\alpha G^{(p)} + \beta G^{(s)}] \quad (9)$$

where α and β are not equal to zero simultaneously.

For the asymptotic investigation of the roots of the equation $\Delta(k, \sigma) = 0$ we will need the asymptotic expression for the solutions of the system (4) when $k \to +\infty$. It is convenient to write (4) down as a system of four equations of the first order. We set

$$G_x = z_1, \qquad G_z = z_2, \qquad \frac{d}{dz} G_x = ikz_3, \qquad \frac{d}{dz} G_z = ikz_4 \quad (10)$$

The system (4) is then reduced to the system

$$\mathbf{z}' = ikH(z, \sigma)\,\mathbf{z} + K(z)\,\mathbf{z} \quad (11)$$

for the vector $\mathbf{z} = (z_1, z_2, z_3, z_4)$ (the stroke designates differentiation with respect to z). Here the elements of the fourth-order matrix H depend on $\mu(z)$, $\nu(z)$, $\rho(z)$ and linearly on σ^2, the elements of K depend on $\mu'(z)$, $\nu'(z)$ and $\rho'(z)$.

With σ fixed, one can apply the classical theorem on the asymptotic expansion of solutions of linear systems of ordinary differential equations containing a large parameter (see, for instance, [1,2]). In accordance with the conditions of that theorem we will require that the elements of the matrix H be differentiable twice (the elements of the matrix K, once) with respect to z on some finite interval $[0, \zeta]$. If for $0 \leqslant z \leqslant \zeta$ the characteristic numbers $\lambda_j (j = 1, 2, 3, 4)$ of the matrix H are distinct and for any fixed pair of indices j, l the quantities $\operatorname{Re}[ik(\lambda_j - \lambda_l)]$ do not change sign, then for a sufficiently large k there exists in the interval $[0, \zeta]$ the fundamental matrix $\Phi(z, k, \sigma)$ of the system (11) such that

$$\Phi(z, k, \sigma) = \left[\Phi_0(z, \sigma) + \frac{\Phi_1(z, \sigma)}{(ik)} + O(k^{-2})\right] \exp \int_0^z [ik\Lambda(\xi, \sigma) + Q(\xi, \sigma)]\,d\xi \quad (12)$$

Here Φ_0 is a non-singular matrix which reduces H to the diagonal form Λ

$$\Phi_0^{-1} H \Phi_0 = \Lambda, \qquad \Phi_1 = \Phi_0 V, \qquad T = \Phi_0^{-1} K \Phi_0 - \Phi_0^{-1} \Phi_0'$$

$$V_{lj} = \frac{T_{lj}}{\lambda_j - \lambda_l} \quad \text{for } j \neq l, \qquad V_{jj} = \int_0^z \sum_{l \neq j} T_{jl} V_{lj}\,d\xi$$

Q is a diagonal matrix with elements $Q_{jj} = T_{jj}$. The estimate in formula (12) is uniform with respect to $z \in [0, \zeta]$.

It is easy to see that the theorem remains valid also in the case under consideration, if the parameter σ varies in some bounded region S in which the conditions of the theorem are satisfied. Here the estimate in formula (12) is uniform with respect to $\sigma \in S$. (The region S depends, generally speaking, on the choice of ζ.)

It can be readily calculated that in the case under consideration we can take

$$\lambda_1 = im_p(z, \sigma), \qquad \lambda_2 = -im_p(z, \sigma), \qquad \lambda_3 = im_s(z, \sigma), \qquad \lambda_4 = -im_s(z, \sigma)$$

where

$$m_p^2(z, \sigma) = 1 - \sigma^2 n_p^2(z), \qquad m_s^2(z, \sigma) = 1 - \sigma^2 n_s^2(z)$$
$$n_p^2(z) = \rho(z)/\nu(z), \qquad n_s^2(z) = \rho(z)/\mu(z)$$

In the plane σ let us make the branch cut $(-\infty, -v_s(z)]$ and $[v_s(z), +\infty)$ and let us fix the branches of the double-valued functions $m_p(z, \sigma)$ and $m_s(z, \sigma)$ by the condition $m_p(z, 0) = m_s(z, 0) = +1$. Let $\lambda(z)$, $\mu(z)$ and $\rho(z)$ be twice continuously differentiable functions. Obviously, the velocity $v_p(z)$ of the longitudinal wave and the velocity $v_s(z)$ of the transverse wave and their reciprocal values $n_p(z) = v_p^{-1}(z)$ and $n_s(z) = v_s^{-1}(z)$ also have continuous second derivatives. Suppose $\min v_s(z) = \omega$ for $z \in [0, \zeta]$. It is easy to see that the region S can be chosen in such a manner that, for $0 < \varepsilon < 1/2\ \omega$, the interval $[\varepsilon, \omega - \varepsilon]$ together with a portion of its neighborhood falls entirely into S.

Out of four linearly independent vectors, which are the solutions forming the fundamental matrix of the system (11), two vectors (we will call them $\mathbf{z}^{(p)}$ and $\mathbf{z}^{(s)}$) have in the domain S the property of decreasing with depth, the other two of increasing. Correspondingly, there are (for sufficiently large k) two decreasing solutions $\mathbf{G}^{(p)}$ and $\mathbf{G}^{(s)}$ of the system (4). According to formulas (10), (12), (7) and (8) we have, for $k \to +\infty$

$$\Delta(k, \sigma) = (ik)^2 \Delta_0(\sigma) + ik\Delta_1(\sigma) + O(1) \tag{13}$$

uniformly in S.

It is well known [1], that for fixed $k > 0$ and z the functions $\mathbf{z}^{(p)}(z, k, \sigma)$ and $\mathbf{z}^{(s)}(z, k, \sigma)$, and, hence, also $\Delta(k, \sigma)$ are regular with respect to σ (at least in the domain S). It turns out that, up to an inessential factor, $\Delta_0(\sigma)$ coincides with the well known Rayleigh's determinant

$$R_0(\sigma) = [1 + m_s^2(0, \sigma)]^2 - 4m_s(0, \sigma)\, m_p(0, \sigma)$$

which has a positive root $v_0 < v_s(0)$, and also $R_0'(v_0) \neq 0$. The functions $\Delta_0(\sigma)$ and $\Delta_1(\sigma)$ are regular in the domain S. Hence* if $v_0 \in S$, then for sufficiently large k there exists a root $v_R(k)$ of the equation $\Delta(k, \sigma)$ such that

$$v_R(k) = v_0 + k^{-1}v_1 + O(k^{-2}) \tag{14}$$

In order to have $v_0 \in S$, it is sufficient to require that $v_0 < \omega$. On sufficiently small intervals $[0, \zeta]$ this condition is always satisfied, since $v_0 < v_s(0)$ and the function $v_s(z)$ is continuous. However, if at a certain depth the velocity $v_s(z)$ of the transverse wave becomes equal to v_0, this condition will be violated and the asymptotic formula (12) will become invalid in the neighborhood of the point $\sigma = v_0$. (The case in which the condition $v_0 < v_s(z)$ is violated is not considered here.)

The correction v_1 can be easily computed by means of the formula

$$v_1 = i\,\frac{\Delta_1(v_0)}{\Delta_0'(v_0)}$$

Namely

$$v_1 = v_0\,\frac{f}{g} \tag{15}$$

Here

$$g = 2r\left(r - 2 + \frac{m_p}{m_s} + p\,\frac{m_s}{m_p}\right) > 0$$

$$f = (m_s + m_p)\left(\frac{4}{r}\,\frac{\mu'}{\mu} - \frac{\rho'}{\rho}\right) + \frac{rm_p}{2(1-r)}\left(\frac{\mu'}{\mu} - \frac{\rho'}{\rho}\right) +$$

$$+ \frac{prm_s}{2(1-pr)}\left(\frac{\nu'}{\nu} - \frac{\rho'}{\rho}\right) - \frac{2(2-r)}{m_s + m_p}\left[\frac{\mu'}{\mu} - \frac{\rho'}{\rho} + \frac{(2-r)(4-r)}{2r}\,\frac{\mu'}{\mu}\right]$$

$$m_s = \sqrt{1-r}, \qquad m_p = \sqrt{1-pr}, \qquad r = \frac{v_0^2}{v_s^2(0)} < 1, \qquad m_s m_p = \left(1 - \frac{r}{2}\right)^2$$

The coefficients f and g are real; the sign of f depends on the relation of quantities

$$\frac{\mu'(0)}{\mu(0)}, \quad \frac{\nu'(0)}{\nu(0)}, \quad \frac{\rho'(0)}{\rho(0)}$$

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* It is easy to show that, if $\varphi(z)$ and $\psi(z)$ are regular in the neighborhood D of a simple root z_0 of the function $\varphi(z)$, then for sufficiently small $|q|$ the equation $\varphi(z) + q\psi(z) = 0$ has in D a root $z_*(q)$ such that

$$z_*(q) - z_0 = -q\,\frac{\psi(z_0)}{\varphi'(z_0)} + O(q^{-2})$$

and may vary in different cases.

For instance, if v_p and v_s are constant, so that $\lambda(z)$, $\mu(z)$ and $\rho(z)$ are proportional, then

$$v_1 = v_0\rho'(0)\, c_1, \qquad c_1 > 0$$

if λ and μ are constant, $\rho(z)$ is arbitrary, then

$$v_1 = -v_0\rho'(0)\, c_2, \qquad c_2 > 0$$

Thus, if coefficients of the equations of elasticity are sufficiently smooth, then in any interval $[0, \zeta]$ in which $v_s(z) > v_0$, for sufficiently large k, there exists a solution of the problem (1) to (2) which has the form (9), with $\sigma_*(k) = v_R(k)$. By virtue of formulas (14), (12) and (10) this solution for $k \to +\infty$ has the asymptotic expansion

$$\mathbf{u}_R(x, z, t; k) = \tag{16}$$
$$= \exp\,[ik(x - v_0 t)] \exp(-itv_1) \exp\left(-k \int_0^z \sqrt{1 - v_0^2 n_s^{-2}(\xi)\, d\xi}\right)[\mathbf{F}(z) + O(k^{-1})]$$

and, hence, allows the estimate

$$u_R(x, z, t; k) = O(e^{-kzc_3}), \qquad c_3 > 0 \tag{17}$$

Here, $\mathbf{F}(z)$ is a smooth vector; both estimates are uniform with respect to z in the interval $[0, \zeta]$.

Thus, the nonhomogeneity of the medium causes an additional factor $\exp(-itv_1)$, which depends on the values of gradients of λ, μ, ρ on the boundary, to appear in the asymptotic expansion of the solution. Since the factor v_1 is real, it does not change the amplitude of the Rayleigh wave in its propagation along the boundary of the half-space.

The Rayleigh waves in nonhomogeneous media of special types have been studied before by the method of exact solutions in [3,4]. In those papers the relations of type (14) of the present article have been obtained, as well as some more refined effects: the change of amplitude of a wave as it is propagated along the surface in the case when the condition $v_s(z) > v_0$ is violated, the absence of surface waves for small frequencies.

In conclusion it is necessary to note that the non-steady-state Rayleigh waves in a general case of a nonhomogeneous elastic body with a surface of an arbitrary shape have been studied by the radial method [5]. According to the formulas of the radial method, the nonhomogeneity with respect to depth causes an additional factor e^{-itv_1} to appear in the expression for the intensity of the wave front. A comparison of values of the quantity v_1 in the particular case $\lambda(z)/\lambda(0) = \mu(z)/\mu(0) = \rho(z)/\rho(0)$, computed by the formulas of [5] and by formula (15) of the present paper, has lead to identical results in both methods.

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